XXIII. *The Principles and Illuſtration of an advantageous Method of arranging the Differences of Logarithms, on lines graduated for the Purpoſe of Computation. By Mr.* William Nicholſon; *communicated by Sir* Joſeph Banks, *Bart. P. R. S.*

Read March 29, 1787.

1. IF two geometrical ſeries of numbers, having the ſame common ratio, be placed in order with the terms oppoſite each other; the ratio, between any term in one ſeries and its oppoſite in the other, will be conſtant [*].

2. And likewiſe the ratio of a term in one ſeries to any term in the other, will be the ſame as obtains between any other two terms having the ſame relative poſition and diſtance [†].

3. In all ſuch pairs of geometrical ſeries, as have the ſame common ratio, the laſt mentioned property obtains, though the firſt antecedent and conſequent be taken in one pair, and the ſecond in any other pair [‡].

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[*] Geom. ſeries $\begin{cases} a \quad an \quad an^2 \quad an^3 \quad an^4 \\ b \quad bn \quad bn^2 \quad bn^3 \quad bn^4 \end{cases}$

Then $a : b :: an : bn :: an^2 : bn^2$, &c.

[†] In the foregoing ſeries $a : bn^2 :: an^2 : bn^4 :: an : bn^3$, &c.

[‡] Geo. ſeries $\begin{cases} a \quad an \quad an^2 \quad an^3 \quad an^4 \\ b \quad bn \quad bn^2 \quad bn^3 \quad bn^4 \end{cases}$

Geo. ſeries $\begin{cases} d \quad dn \quad dn^2 \quad dn^3 \quad dn^4 \\ \frac{bd}{a} \quad \frac{bdn}{a} \quad \frac{bdn^2}{a} \quad \frac{bdn^3}{a} \quad \frac{bdn^4}{a} \end{cases}$

Then $a : bn^2 :: dn : \frac{bdn^3}{a}$, &c.

4. If the differences of the logarithms of numbers be laid in order upon an arrangement of equi-diſtant parallel right lines, in ſuch a manner as that a right line, drawn acroſs the whole, ſhall interſect it at diviſions which denote numbers in geometrical progreſſion; then, from the condition of the arrangement and the property of this logarithmic line, it follows, firſt, that every right line, ſo drawn, will, by its interſections, indicate a geometrical ſeries of numbers*; ſecondly,

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* Let AB, CD, EF (Tab. X. fig. 5.) be portions of the logarithmic line, arranged according to the condition: let GH be a right line drawn acroſs, ſo as to paſs through points of diviſion *e*, *c*, *a*, denoting numbers in geometrical progreſſion: then will any other line IK, drawn acroſs the arrangement, alſo paſs through points *f*, *d*, *b*, denoting numbers in geometrical progreſſion.

Demonſtration. From one of the extreme points of interſection *f* in the laſt named line IK draw the right line *fg*, parallel to GH, and interſecting the arrangement in the points *i*, *h*; and the ratios of the numbers $e : f$, $c : i$, and $a : h$, will be equal, becauſe the intervals on the logarithmic line, or differences of the logarithms of thoſe numbers, are equal:

Or $\frac{e}{c}=\frac{f}{i}$ and $\frac{c}{a}=\frac{i}{h}$.

But $\frac{e}{c}=\frac{c}{a}$ by the condition.

Therefore $\frac{f}{i}=\frac{i}{h}$; or the numbers *f*, *i*, *h*, are in the ſame continued ratio as the numbers *e*, *c*, *a*.

Again, the point *f*, the line *id*, and the line *hb*, are in arithmetical progreſſion, and denote the differences of the logarithms of the numbers *f* and *f*, *i* and *d*, *h* and *b*.

The quotients of the numbers themſelves are therefore in geometrical progreſſion, that is,

$\frac{f}{f} : \frac{d}{i} : \frac{b}{h}$, or $\frac{i}{d}=\frac{dh}{bi}$.

Or $\frac{i}{d}=\frac{di}{bf}$, by ſubſtituting $\frac{i}{f}$ for its equal $\frac{h}{i}$.

Whence $\frac{f}{d}=\frac{d}{b}$ or $f : d : b$. Q. E. D.

that ſuch ſeries, as are ſo indicated by parallel right lines, will have the ſame common ratio*; and, thirdly, that the ſeries thus indicated by two parallel right lines, ſuppoſed to move laterally without changing either their mutual diſtance, or parallelifm to themſelves, will have each the ſame common ratio; and in all pairs of ſeries indicated by ſuch two lines, the ratio between an antecedent on one parallel and the oppoſite term on the other, taken as a conſequent, will be conſtant†.

5. Thus far the logarithmic line has been conſidered as unlimited. If, therefore, an antecedent and conſequent be given, it will be poſſible to find both on the arrangement, and to draw two parallel lines, one over each number: and if the lines be then ſuppoſed to move, without changing either their diſtance or abſolute direction, ſo that the line, which before marked an antecedent, may now mark a new antecedent; the other (by 2. and 3.) will mark a number, at the ſame relative poſition and diſtance, which ſhall be the conſequent to this laſt antecedent after the ſame ratio.

6. Suppoſe a logarithmic line to contain no more than a ſingle range of numbers from 1 to 10, it will not be neceſſary, for the purpoſes of computation, to repeat it; for if a ſlider

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* In the ſame manner as it was proved that the line fg, parallel to GH, paſſes through points of diviſion, denoting numbers in the ſame continued ratio as thoſe indicated by the line GH, it may alſo be ſhewn, that the line LM parallel to any other line IK, will paſs through a ſeries of numeral points, having the ſame continued ratio as the ſeries indicated by that line IK to which it is parallel.

† Becauſe the lines preſerve their paralleliſm to their former ſituation, they will indicate geometrical ſeries having the ſame common ratio as before; and, becauſe their diſtance meaſured on the logarithmic line remains unchanged, the differences of the logarithms of oppoſite numbers, and conſequently their ratio, will be conſtant.

or beam have two fixed points at the diſtance of the interval between 1 and 10, and a moveable point be made to range between theſe (always to indicate the antecedent); then, I ſay, if the conſequent fixed point fall without the rule, the other fixed point will ſhew the diviſion it would have fallen on, if the rule had been prolonged. This may be eaſily applied to the arrangement deſcribed, N° 4.

7. If the arrangement conſiſt only of the logarithms from 1 to 10, and the parallel croſs lines interſect that geometrical ſeries whoſe ſucceſſive ratios altogether, with that of the laſt to the firſt, make by compoſition the ratio $\frac{1}{10}$, the contrivance, N° 6, may be applied to ſhew ſuch conſequents as fall, laterally, without the rule.

8. It is convenient that the arrangement of the lines be diſpoſed ſo as to occupy a rectangular parallelogram; or, in other words, that the croſs line, cutting the ſeries laſt mentioned, may be at right angles to the length of the rule.

The conſtruction of an inſtrument on the foregoing principles admits of various diſpoſitions of the graduated lines and apparatus for meaſuring intervals upon them. Fig. 1. is a rule conſiſting of ten parallel lines, equivalent to a double line of numbers upwards of 20 feet in length. Fig. 2. is a beam compaſs for meaſuring intervals. The parts B, A, C, apply to the ſurface of the rule; the middle A being moveable ſideways in a groove in the piece DE, ſo as always to preſerve its paralleliſm to the external pieces B, C, which are fixed at a diſtance equal to the length of the rule, and have their edges placed according to the condition in § 7. which is here at right angles to the length. The piece DE, in the uſe, is applied to the edge FG of the rule. The edges or borders H, I, K, L, may

may be made of tranſparent horn or tortoiſe-ſhell, which a
in many reſpects more convenient than any opake ſubſtance.

The Uſe. Apply the edge of either B or C to the conſequen
and ſlide the piece A to the antecedent, obſerving the differen
between the numbers on the pieces denoting the lines they a
found on; then, if the ſame edge of A be applied to any oth
antecedent, the other piece B or C, made uſe of, will inte
ſect a conſequent in the ſame ratio upon that line of the a
rangement, which has the ſame ſituation, with reſpect to tl
antecedent, as the line of the former conſequent had to its ant
cedent. The numbers on the pieces ſerve to indicate the rel
tive ſituations. But if B be the conſequent piece, and fa
without the rule, the piece C will ſhew the conſequent one li
lower; or if C, in the like caſe, fall without the rule, B w
ſhew the conſequent one line higher. It would be eaſy to mal
the ſame kind of proviſion for the numbers which fall lateral
without the rule: and it might be found convenient if, for tl
purpoſe of computation, inſtruments of this kind were to l
made with an hundred or more lines. But in the preſent i
ſtrument, the numbers on the pieces will anſwer the ſan
purpoſe: for if a conſequent fall on a line at any given nun
ber of intervals without the rule, it will be found on that li
of the arrangement, which occupies the ſame number of i
tervals, reckoned inwards from the oppoſite edge of the rule.

Fig. 3. is a Gunter's ſcale, equivalent to that of 28
inches in length, publiſhed by the late Mr. Robertson.
is, however, but one-fourth of the length, and contai
only one-fourth of the quantity of diviſion. In the ſlid
GH is a moveable piece AB, acroſs which a fine line
drawn; and there are alſo lines CD, EF, drawn acroſs tl
ſlider, at a diſtance from each other equal to the length of tl
rule. The uſe of this is ſimilar to that of the foregoing. Tl

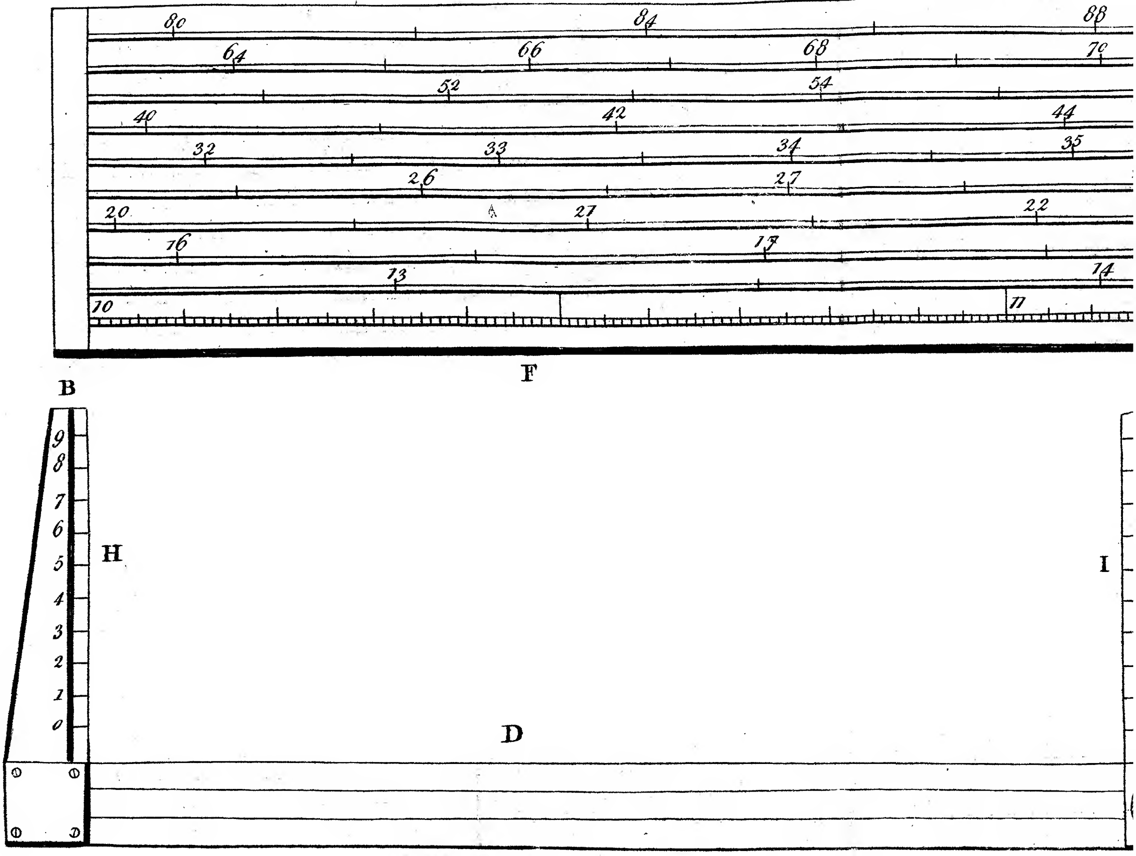
line CD or EF is to be placed at the conſequent, and the line in the piece AB at the antecedent: then, if the piece AB be placed at any other antecedent, the ſame line CD or EF will indicate its conſequent in the ſame ratio taken the ſame way; that is to ſay, if the antecedent and the conſequent lie on the ſame ſide of the ſlider, all other antecedents and conſequents in that ratio will lie in the ſame manner; and the contrary if they do not, &c. But if the conſequent line fall without the rule, the other fixed line on the ſlider will ſhew the conſequent; but on the contrary ſide of the ſlider to that where it would elſe have been ſeen by means of the firſt conſequent line.

Fig. 4. is an inſtrument equivalent to the ſame rule of 28½ inches long. It conſiſts of three concentric circles engraved and graduated on a plate of about 1½ inch in diameter. From the center proceed two legs A, B, having right-lined edges in the direction of radii. They are moveable either ſingly or together. To uſe this inſtrument, place one of the edges at the antecedent, and the other at the conſequent, and fix them to that angle. The two legs being then moved together, and the antecedent leg placed at any other number, the other leg will give its conſequent in the like poſition or ſituation on the lines. If the line CD happen to lie between the legs, and B be the conſequent leg, the number ſought will be found one line farther from the center than it would otherwiſe have been; and, on the contrary, it will be found one line nearer in the like caſe, if A be the conſequent leg. This inſtrument, differing from fig. 1. only in its circular form, and the advantages reſulting from that form, the lines muſt be taken to ſucceed each other in the ſame manner laterally; ſo that numbers, which fall either without or within the arrangement of circles, will be found on ſuch lines of the arrangement as would have occupied the vacant places if the ſucceſſion of lines had been indefinitely repeated ſideways.

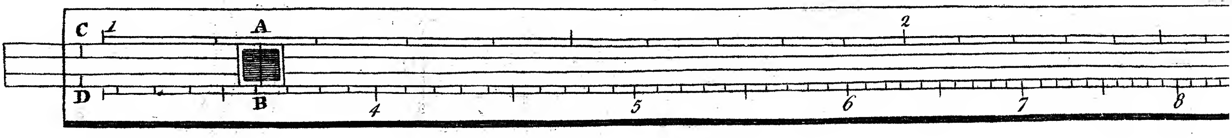
I approve of this conſtruction, as ſuperior to every other which has yet occurred to me, not only in point of convenience, but likewiſe in the probability of being better executed, becauſe ſmall arcs may be graduated with very great accuracy, by diviſions transferred from a larger original. The inſtrument, fig. 1. may be conveniently contained in a circle of about 4½ inches diameter.

The circular inſtrument is a combination of the GUNTER's line and the ſector, with the improvements here pointed out. The property of the ſector may be uſeful in magnifying the differences of the logarithms in the upper part of the line of ſines, the middle of the tangents, or the beginning of the verſed ſines. It is even poſſible, as mathematicians will eaſily conceive, to draw ſpirals on which graduations of parts, every where equal to each other, will ſhew the ratios of thoſe lines by means of moveable radii ſimilar to thoſe in this inſtrument. But I do not, in this Diſcourſe, propoſe to enter into enquiries reſpecting the nature of ſuch curves, nor their utility in the preſent buſineſs.

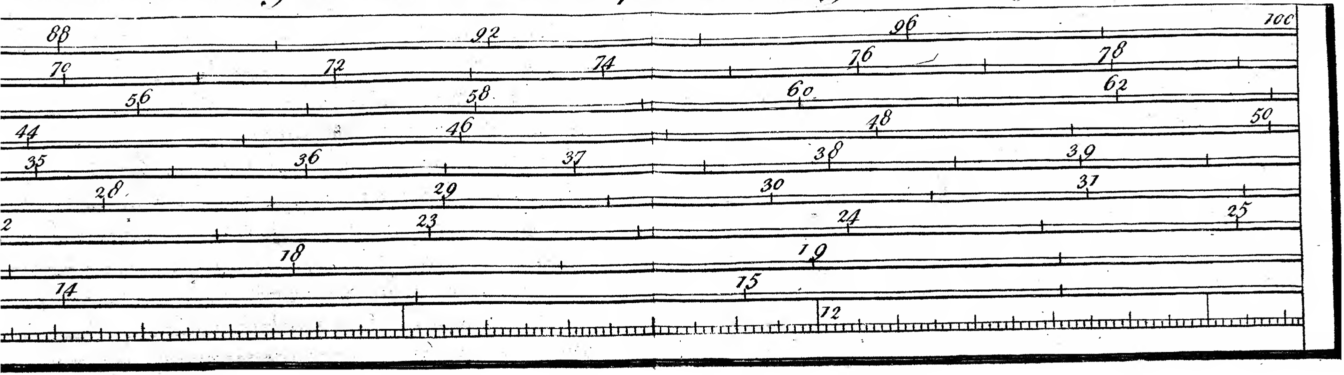
Sketch of an Instrument, equivalent to a



Gunter's Scale, equivalent to those commonly made of 28½ Inches



to a common Gunter's Scale 20 feet in length. Fig. 1.



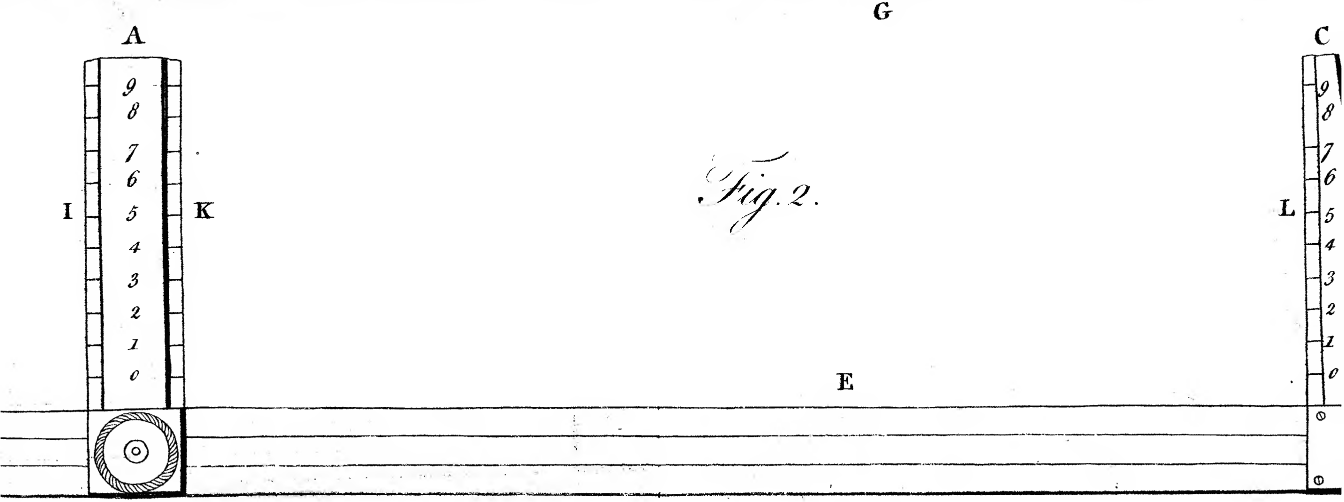
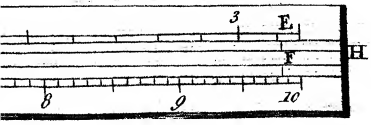
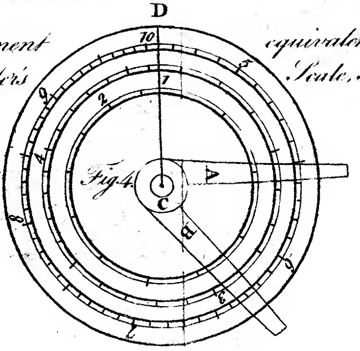


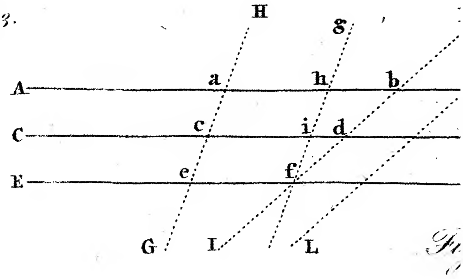
Fig. 2.

ches long. Fig. 3.



Instrument equivalent to the Gunter's Scale, Fig. 3.





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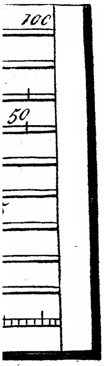
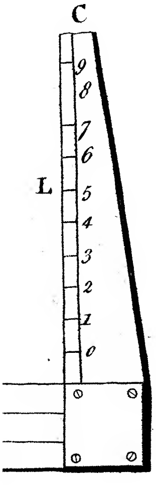




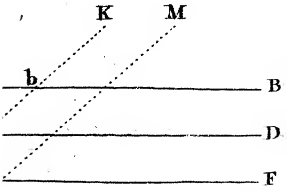
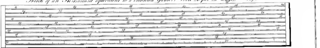


Fig. 5.

Sketch of an Instrument equivalent to a common Gunter's Scale 20 feet in length Fig 1



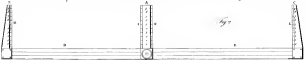
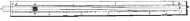
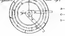


Fig 2

Gunters Scale equivalent to those commonly made of 20½ Inches long Fig 3



Instrument equivalent to the Gunters Scale Fig 3



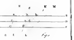


Fig 5